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General-Relativistic Curvature of Pulsar Vortex Structure.

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Abstract — The motion of a neutron superfluid condensate in a pulsar is studied. Several theorems of general-relativistic hydrodynamics are proved for a superfluid. The average density distribution of vortex lines in pulsars and their general-relativistic curvature are derived.

Key words: pulsars

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1 Introduction.

The superfluidity of matter in rotating neutron stars combines with the closeness of their gravitational ($r_g = 2GM/c^2$) and geometrical (R) radii, where (G) is the gravitational constant, (c) is the speed of light, and (M) is the neutron-star mass (see, e.g., Manchester and Taylor 1980). Therefore, general-relativistic effects can appreciably affect the processes in the superfluid cores of pulsars and the mechanisms of glitches. In this regard, Andreev et al. (1995) discussed low angular velocities in general relativity: $\Omega < \Omega_c < \hbar/(m^* R^2)$, where m^* is the mass of a superfluid condensate particle (a Cooper pair), $\Omega \sim \Omega_c$ (see Kirzhnits and Yudin 1995), and the number of vortex lines (VLs) is 0 and 1, respectively.

Here, we deal with the realistic, opposite case,

$$\Omega_c \ll \Omega < c/R, \tag{1}$$

when there is a dense system of VLs. Recently, the interest in this problem has risen dramatically, which is most likely attributable to an increase in the accuracy of measuring glitches in pulsars. Consequently, it becomes possible to detect in principle post-Newton gravimagnetic effects in pulsars, which are described below. See, e.g., Prix (2000) and Langlois (2000) for an overview of

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this subject. Here, we propose a method of solving the problem in question that slightly differs from that proposed by these authors.

Bekarevich and Khalatnikov (1961) proved that rigid-body rotation is stable in the nonrelativistic case, and that VLs are rectilinear and distributed at a constant density. Because of general-relativistic gravielectric and gravimagnetic effects, which are determined, respectively, by $\nabla g_{\alpha\beta}$ and $\operatorname{rot}\mathbf{g}$ (by the definition of Landau and Lifshitz (1988b), $\mathbf{g} = \{-g_{0\alpha}/g_{00}\}$ and g_{ik} is the metric tensor), differential rotation becomes stable (to be more precise, this rotation is stable dynamically, while rigid-body rotation is stable kinematically as before; see below). In this case, VLs curve and are redistributed in space. These effects are described below.

2 Analyzing changes in the system whenan ordinary fluid is replaced with a quantum, fermi superfluid.

2.1 General Relations for a Superfluid Condensate in General Relativity

The wave function of a superfluid condensate is known (see, e.g., Bekarevich and Khalatnikov 1961) to be

$$\psi = \nu \exp(i\phi).$$

Given the identity $u_i u^i = 1$ [2], the generalization of the nonrelativistic relation $\mathbf{v} = \frac{\hbar}{m^*}\nabla\phi$ to general relativity is

$$u_i = \frac{\partial_i \phi}{k}, \qquad k^2 = \partial_i\phi\partial^i\phi. \tag{2}$$

Indeed, let us derive an expression for a general-relativistic superfluid current $j_i = n u_i$ [3] with the continuity equation $j_i^{;i} = j^i_{;i} = 0$.

Let us write the Lagrangian of a superfluid condensate in general relativity by using the Madelung hydrodynamic representation (see Grib et al. 1980; Bogolyubov and Shirkov 1993):

$$L = \frac{1}{2}\psi^*_{;i}\psi^{;i} + |\psi|^2\tilde{F}(|\psi|^2) = \frac{1}{2}\nu^2\partial_i\phi\partial^i\phi + \nu^2 F(\nu^2). \tag{3}$$

[2] The indices i, j, k, l, m, n run the series 0, 1, 2, 3; the indices α, β, γ run the series 1, 2, 3.

[3] We introduce the notation in which the scalar densities ε, p, $\omega = p + \varepsilon$, and n are identified with their eigenvalues (i.e., with the values in a commoving frame of reference), respectively: the energy density, pressure, thermal function, and density of the particles; u_i are the components of the 4-velocity vector for the matter.

The principle of least action relative to a change in ϕ yields

$$(\nu^2 \partial^i \phi)_{;i} = 0. \quad (4)$$

Consequently, the current vector is $j^i = \nu^i = const \cdot \nu^2 \partial^i \phi$. This is seen from a comparison of j^i and j^i_{sf} in the nonrelativistic case: $u^i = v^i/c$ and $j^i = nv^i/c$. For a superfluid, we write in this case: $n = \nu^2$, $v^i = \frac{\hbar}{m^*}\partial^i \phi$, and $j^i_{sf} = \frac{\hbar}{m^* c}\nu^2 \partial^i \phi$. Since the continuity equation $j^i_{;i} = 0$ must hold in any case, we derive Eq. (4) for a superfluid. Hence, we obtain for a superfluid current in general relativity

$$j^i = \frac{\hbar}{m^* c}\nu^2 \partial^i \phi = nu^i. \quad (5)$$

or, for covariant quantities,

$$j_i = \frac{\hbar}{m^* c}\nu^2 \partial_i \phi = nu_i. \quad (6)$$

A scalar multiplication of the latter expression by Eq. (5) yields

$$\left(\frac{\hbar}{m^* c}\right)^2 \nu^4 \partial^i \phi \partial_i \phi = n^2,$$

Expression (2) follows from this.

In the Newton approximation, $k^2 \approx (m^* c/\hbar)^2$. The scalar k is identified with $w/(nc\hbar)$ (see below). Hence, since $\nu^2 = n$ in the Newton approximation, Eq. (6) or (2) yields

$$u_i \approx \frac{\hbar}{m^* c}\partial_i \phi. \quad (7)$$

In seeking to approach an optimum regime, a superfluid current undergoes a well-known rearrangement (see Bekarevich and Khalatnikov 1961). As a result, while the current remains potential "almost everywhere", vorticity arises, which generally coincides with the vorticity for an ordinary (nonsuperfluid) fluid at the same point. This occurs, because a system of VLs is formed, with the phase of the wave function $\phi = \varphi + f(t)$ corresponding to each of them [4]. According to (7), we then have for a single vortex

$$u_\varphi \approx \frac{\hbar}{m^* c}. \quad (8)$$

[4] In what follows, we use a cylindrical coordinate system with the z axis directed along the spin axis: $x^{0,1,2,3} = ct, \rho, \phi$, and z.

On the vortex axis itself, where ϕ is uncertain, the wave function loses its meaning, and the superfluidity vanishes. This corresponds to a physically non-superfluid VL "core" of a macroscopically small radius, which is responsible for the nonzero curl of velocity (see Bekarevich and Khalatnikov 1961).

The above reasoning is universal and equally applies to the nonrelativistic and general-relativistic cases.

2.2 The General-Relativistic Bernoulli Theorem

For the subsequent analysis of a superfluid in general relativity, we need the general-relativistic Bernoulli theorem.

The classical Bernoulli theorem is derived for a steady, isentropic fluid flow from the Euler hydrodynamic equations (see Landau and Lifshitz 1988a):

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\nabla \frac{w}{mn}.$$

In the relativistic case, these equations can be written as

$$u^i \partial_i (k^* u_j) = \partial_j k^*, \tag{9}$$

where $k^* = \omega/n$. Note that these equations have a particular solution of the form $k^* u_j = const \cdot \partial_j \phi$, where ϕ is a scalar function of the coordinates. Therefore, in contrast to the nonrelativistic case, the quantity $k^* u_j$ rather than the velocity has the potential. Hence, according to (2), we immediately identify k^* for a superfluid with $c\hbar k$:

$$k^* = \frac{w}{n} = \hbar c k \tag{10}$$

In general relativity, ∂_j is replaced with the covariant derivative ∇_j:

$$u^i \nabla_i (k^* u_j) = \nabla_j k^*, \tag{11}$$

or, expanding the covariant derivative,

$$u^i \partial_i (k^* u_j) - u^i{}^{m}_{ij} k^* u_m = \partial_j k^*.$$

Substituting the expressions for the Christoffel symbols in this relation,

$${}^{m}_{ij} = \frac{1}{2} g^{mn} (\partial_i g_{jn} + \partial_j g_{in} - \partial_n g_{ij}), \tag{12}$$

yields

$$u^i \partial_i (k^* u_j) - \frac{1}{2} k^* u^i u^n \partial_j g_{in} = \partial_j k^*. \tag{13}$$

Denoting

$$\phi_j = \partial_j \phi = k u_j; \quad \phi^j = \partial^j \phi = k u^j, \tag{14}$$

we obtain for a superfluid

$$\phi^i \partial_i \phi_j - \frac{1}{2} \phi^i \phi^n \partial_j g_{in} = \frac{1}{2} \partial_j k^2. \tag{15}$$

In that case, since $k^2 = g_{in} \phi^i \phi^n$, we derive two equations,

$$\phi^\alpha [\partial_\alpha \phi_0 - \partial_0 \phi_\alpha] = 0. \tag{16}$$

and

$$\phi^\alpha [\partial_\alpha \phi_\gamma - \partial_\gamma \phi_\alpha] + \phi^0 [\partial_0 \phi_\gamma - \partial_\gamma \phi_0] = 0. \tag{17}$$

The former follows from the latter. Denoting

$$\sigma_{\alpha\gamma} = \frac{1}{2\pi} [\partial_\alpha \phi_\gamma - \partial_\gamma \phi_\alpha], \tag{18}$$

we obtain for $\sigma_{\alpha\gamma}$ [5]:

$$\sigma_{\alpha\gamma} = \frac{\phi^0}{2\pi \phi^\beta \phi_\beta} [\phi_\alpha \partial_\gamma \phi_0 - \phi_\gamma \partial_\alpha \phi_0 + \phi_\gamma \partial_0 \phi_\alpha - \phi_\alpha \partial_0 \phi_\gamma]. \tag{19}$$

We emphasize that our system is not axially symmetric because it contains vortices, but this symmetry is restored by averaging over the VL distribution. As a result, there is no steady state in this case. Clearly, an infinite number of stationary frames of references exist for an ordinary fluid uniformly rotating in a vessel. Below, we show that there is a (unique) frame of reference for a rotating superfluid comoving with VL cores in which all quantities are stationary. For this frame of reference, we derive from (17) $\partial_\gamma \phi'_0 = 0$ [6] on all current lines where $\sigma_{\alpha\gamma} = 0$ (∂_γ in any direction). Therefore,

$$\boxed{k u'_0 = \phi'_0 = const. \qquad (20)}$$

[5] Since the condensate ceases to exist on vortex lines, the concept of phase loses its meaning; thus, the theorem on the curl of a gradient ceases to be valid on vortex cores, and the 4-gradient of phase $\partial_i \phi$ is replaced with a 4-vector ϕ_i.

[6] The prime means that the quantities under consideration refer to a stationary frame of reference outside VL cores.

This is the general-relativistic analog of the Bernoulli theorem. In the nonrelativistic limit (see Landau and Lifshitz 1988a):

$$k = \frac{1}{\hbar c}\frac{w}{n} \to \frac{m^* c}{\hbar}(1 + \frac{p}{\tilde{\rho} c^2}), \quad u_0 = \sqrt{\frac{g_{00}}{1 - v^2/c^2}} \to (1 + \frac{\tilde{\varphi}}{c^2})(1 + \frac{v^2}{2c^2}) \to$$

$$\to 1 + \frac{\tilde{\varphi}}{c^2} + \frac{v^2}{2c^2}, \Longrightarrow k u_0 \to \frac{m^* c}{\hbar}(1 + \frac{\tilde{\varphi} + v^2/2 + p/\tilde{\rho}}{c^2}) = const,$$

where $\tilde{\phi}$ and $\tilde{\rho}$ are the Newton gravitational potential and the matter density, respectively. This is the ordinary Bernoulli theorem.

For a rotating superfluid with vortices, the laboratory frame of reference ceases to be stationary, but we can choose a frame of reference for a rotating superfluid that comoves with VL cores; in this case, the frame of reference is stationary and does not comove with the superfluid.

3 Determining the relationship between dynamical superfluid parameters

3.1 The Principle of Least Action for a Rotating Superfluid in General Relativity

Hartl and Sharp (1967) proved that rigid-body rotation with an angular velocity equal to the shell angular velocity is most favorable for an ideal, ordinary fluid in general relativity.

Let us prove that this assertion remains also valid for a superfluid [7].

In view of the general-relativistic Bernoulli theorem and by analogy with the study of Hartl and Sharp (1967) for an ordinary fluid, we assume the spatial components $\partial_\gamma \phi$ and the probability density ν^2 for a condensate particle to be detected at a given point to be independent variables for superfluid dynamics. In addition, we will remember that x_k are the variables that determine the vortex shape and coordinates, while the metric components g_{ik} are the variables that determine the system's gravitational field.

Given the conservation of momentum and the total number of particles, the expression for the total energy of a superfluid can be written as (Hartl and

[7] By the angular velocity of a superfluid, we imply the mean value of $u^\varphi/(cu^0)$ averaged over the surface orthogonal to the vortex direction near the point under consideration whose area is much smaller than the system's cross-sectional area, but, at the same time, is still much larger than the square of the mean separation between vortices. As we show below, this condition can definitely be satisfied for pulsars.

Sharp 1967)

$$\tilde{L} = \nu^2 \phi_0 \phi^0 - \nu^2 \left[F + k^2/2\right] - \Omega(-\frac{1}{c}\nu^2 \phi^0 \phi_\varphi) - \mu\nu^2\phi^0, \tag{21}$$

where Ω and μ are the corresponding Lagrange factors.

In addition, we must take into account the Lagrangian L_0 of the shell. Given momentum conservation, the analog of (21) for the shell can be written as

$$\tilde{L}_o = J_o \Omega_o^2/2 - \Omega J_o \Omega_o,$$

where J_0 and Ω_0 are the moment of inertia and angular velocity of the shell in the frame of reference under consideration, respectively. Varying over Ω_0 leads to the equality

$$\Omega = \Omega_o.$$

If, however, the satisfaction of the general-relativistic potentiality condition for a superfluid flow must also be taken into account, then, according to the general rules for imposing additional conditions, we must add the term $\Delta\tilde{L}$ to $\tilde{L}$ to express the fact that the circulation of the phase gradient over a closed contour is proportional to the number of vortices crossing this contour. This is the general-relativistic generalization of the potentiality condition for a superfluid flow:

$$\oint \partial_\gamma \phi \, dx^\gamma = \int (\partial_\beta \partial_\gamma \phi - \partial_\gamma \partial_\beta \phi) \, dx^\beta \wedge dx^\gamma = 2\pi K, \tag{22}$$

where $dx^\beta \wedge dx^\gamma$ is the directed surface element pulled over contour Γ, and K is the number of vortices crossing the contour. Thus, the general-relativistic potentiality condition, according to (18), can be written as

$$2\pi\sigma_{\beta\gamma} = \sum_{k=1}^{N} 2\pi n^k_{\beta\gamma} \delta^2(x - x_k), \tag{23}$$

while the addition to $\tilde{L}$ corresponding to this condition can be written as

$$\Delta\tilde{L} = \xi^{\beta\gamma}(x)[2\pi\sigma_{\beta\gamma} - \sum_{k=1}^{N} 2\pi n^k_{\beta\gamma} \delta^2(x - x_k)]. \tag{24}$$

Here, according to (18), $2\pi\sigma_{\beta\gamma}$ is the curl of ϕ_γ; $n^k_{\beta\gamma}$ is a unit antisymmetric tensor dual to one of the surfaces, which are orthogonal to the vectors of the vortex direction at each point, $n^k_{\beta\gamma}$ is defined at the point of intersection of this surface with vortex k, and the summation over k is performed over all

vortices in the system [8]; $\delta^2(x - x_k)$ is the bivariate delta function, and x is the radius vector defined on this surface; and the antisymmetric tensor $\xi^{\beta\gamma}(x)$ is the Lagrange functional factor.

4 Determining the Relationship between the Components of the 4-Gradient of Phase for a Superfluid Condensate

Action A in general relativity is known to be related to Lagrangian L by

$$A = \int L\, dX, \quad dX = \sqrt{-g}\, d^4x,$$

where the integration is performed over 4-space; g is the determinant of the metric tensor g_{ik}. We denote the integral of $\tilde{L}$ over space by

$$\tilde{E} = \int \sqrt{-g}\tilde{L}\, d^3x, \qquad (\,\Delta\tilde{E} = \int \sqrt{-g}\Delta\tilde{L}\, d^3x\,). \tag{25}$$

By its physical meaning, $\tilde{E}$ is the system's energy for imposed additional conditions, such as allowance for the conservation of angular momentum, the total number of particles in the system, etc.

We break up the integral over space into an integral over the surface dual to the tensor $n^k_{\beta\gamma}$ and integrals over the lengths of the vortices orthogonally crossing this surface. Because of the presence of delta functions, the following sum remains from $\Delta\tilde{E}$:

$$\Delta\tilde{E} = \int \sqrt{-g}2\xi^{\beta\gamma}(x)(\partial_\beta\phi_\gamma)d^3x + \sum_{k=1}^{N} 2\pi l_k\sqrt{-g}\xi^{\beta\gamma}(x_k)n^k_{\beta\gamma},$$

where l_k is the length of vortex line k.

After varying $\Delta\tilde{E}$ over ϕ_γ, the following term remains:

$$-2\partial_\beta(\sqrt{-g}\xi^{\beta\gamma}) \tag{26}$$

If we discard addition (26), then the subsequent analysis will be valid only for an averaged description of the superfluid motion. Let us prove that term (26) vanishes on vortex cores. Since the coordinates x_k must correspond to equilibrium vortex positions in the system, the system must be stable against core displacements orthogonal to the direction of the vortices themselves:

$$2\pi l_k n^k_{\beta\gamma}[\partial_\beta(\sqrt{-g}\xi^{\beta\gamma})]|_{x=x_k} = 0. \tag{27}$$

[8] In general, addition (24) to $\tilde{L}$ should be written for each such surface, but this is implied by default.

This proves the above assertion. The corrections related to the term $\Delta\tilde{E}$ in action will not be considered everywhere, because we are interested in the system's dynamics only when expression (27) holds, i.e., on vortex cores, or, equivalently, only an averaged description of all quantities for the system. Below, we thus denote average quantities by a hat above the symbol (e.g., $\hat{A}$). According to Eq. (3), the principle of least action relative to a change in ν^2 leads to the equation

$$\frac{\partial L}{\partial \nu^2} = \frac{\partial}{\partial \nu^2}\{\nu^2\,[k^2/2 + F(\nu^2)]\} = 0. \qquad (28)$$

Given that $k^2 = g^{\alpha\beta}\phi_\alpha\phi_\beta + 2g^{0\alpha}\phi_0\phi_\alpha + g^{00}(\phi_0)^2$ and denoting

$$A^\alpha = \frac{\partial \phi_0}{\partial \phi_\alpha}, \qquad B^\alpha = \frac{\partial \phi^0}{\partial \phi_\alpha}, \qquad (29)$$

we obtain

$$\frac{\partial k^2}{\partial \phi_\alpha} = 2g^{\alpha\beta}\phi_\beta + 2g^{0\alpha}\phi_0 + 2g^{0\alpha}\phi_\alpha A^\alpha + 2g^{00}\phi_0 A^\alpha = 2[\phi^\alpha + \phi^0 A^\alpha]. \qquad (30)$$

Taking a variational derivative of Eq. (21) with respect to ϕ_γ and ν^2, using Eqs. (28), (29), (30), we derive for the average quantities

$$\hat{B}^\gamma(\hat{\phi}_0 + \frac{\Omega}{c}\hat{\phi}_\varphi - \mu) = \hat{\phi}^\gamma - \frac{\Omega}{c}\hat{\phi}^0\delta^\gamma_\varphi. \qquad (31)$$

$$\hat{\phi}_0 + \frac{\Omega}{c}\hat{\phi}_\varphi - \mu = 0. \qquad (32)$$

Comparing Eqs. (31) and (32), we obtain the analog of rigid-body rotation for a superfluid:

$$\boxed{\frac{\phi^\gamma}{\phi^0}\Big|_{x=x_k} = \frac{\hat{\phi}^\gamma}{\hat{\phi}^0} = \frac{d\hat{x}^\gamma}{dx^0} = \frac{\Omega}{c}\delta^\gamma_\varphi. \qquad (33)}$$

4.1 Magnus Force and the General-Relativistic Theorem on the Conservation of Circulation

The Magnus force acts on a rotating body in an incoming flow and is attributable to a nonzero pressure difference for the opposite sides of the flow

around the body. In tern, the pressure along the body boundary changes because of the Bernoulli theorem: the velocity of the medium that flows around a rotating body changes when going around the body axis.

Let us write the Bernoulli equation for the nonrelativistic case:

$$p + mn\mathbf{V}^2/2 = const \tag{34}$$

It would be natural to consider a portion of the VL core as the body on which the Magnus force acts. As we show below, the Magnus force does not depend on the radius a of this core. Choose a cylindrical coordinate system whose z axis coincides with the rotation axis of this core. Since the core radius a is much smaller than any scales on which the incoming flow produced by the remaining vortices changes appreciably, this flow may be considered constant for the flow around the core. According to (33), the physical velocity of this flow at the core point is

$$\mathbf{V}_0 = [\mathbf{\Omega} \times \mathbf{r}]$$

At the same time, the velocity produced by the core itself on its boundary, according to (8) is

$$\mathbf{V}_k = \frac{\hbar}{m^* a^2}[\mathbf{e}_z \times \mathbf{a}],$$

where $\mathbf{a}$ and $\mathbf{e_z}$ are the vector in the direction of the core radius (equal to a in magnitude) and a unit vector along the core rotation axis, respectively. The total velocity on the core boundary is

$$\mathbf{V} = \mathbf{V}_0 + \frac{\hbar}{m^* a^2}[\mathbf{e}_z \times \mathbf{a}].$$

The force per unit core area is equal to the pressure multiplied by a unit vector: $-\mathbf{e_a} = -\mathbf{a}/a$. Accordingly, the force per unit core length is

$$\mathbf{F} = -\oint \mathbf{e}_a p(\varphi)\, dl = -\oint_0^{2\pi} \mathbf{e}_a p(\varphi) a\, d\varphi. \tag{35}$$

Expressing p from Eq. (34) and substituting it in Eq. (35) yield

$$\mathbf{F} = -\oint_0^{2\pi} \mathbf{a}\{const - mn(\mathbf{V}_0^2 + \mathbf{V}_k^2 + 2\mathbf{V}_0\mathbf{V}_k)/2\}d\varphi. \tag{36}$$

Since the φ — independent terms vanish and since $n \approx \nu^2$ and $m \approx m^*$ for a superfluid in the nonrelativistic case, we derive for the Magnus force per unit

core length

$$\mathbf{F} = \oint\limits_0^{2\pi} \mathbf{a}\{mn\mathbf{V}_0 \frac{\hbar}{m^* a^2}[\mathbf{e}_z \times \mathbf{a}]\} d\varphi = \pi\hbar\nu^2 [\mathbf{V}_0 \times \mathbf{e}_z]. \tag{37}$$

It follows from Eq. (37) that the Magnus force acting on the VL core that is at rest relative to a remote observer causes it to move toward the vessel wall. As it accelerates toward the wall, an incoming flow emerges (to be more precise, the core itself runs on the superfluid); as a result, the Magnus force changes its direction, causing the VL core to precess around some point of the superfluid flow. This precession is rapidly damped, and the core starts moving in such a way that the Magnus force does not act on it, i.e., that the system's energy becomes minimal. This requires that the core be at rest with respect to the superfluid flow at its location. The above reasoning proves that the VL system is frozen in, i.e., that there is no slip in the nonrelativistic case.

In general relativity, expression (34) for the Bernoulli theorem is replaced with expression (20).

For the nonrelativistic case, Hess (1967) showed that the VLs are, as it were, frozen in a superfluid — move at angular velocity Ω, the shell rotation velocity. Thus, there is no slip relative to this angular velocity.

To generalize the slip theorem to general relativity, we do not need to repeat similar calculations in order to determine the Magnus force, which, incidentally, are very complex. It will suffice to note that $\phi_0 = ku_0$, $k = k(|\mathbf{V}|, p, n, g_{ik})$, and $u_0 = u_0(|\mathbf{V}|, g_{ik})$ and that when going around the core of a vortex line, the changes in metric g_{ik} and density n, if any, are so negligibly small [9] that they may be disregarded.

Therefore, we can write for the superfluid portion adjacent to the core

$$p = p(|\mathbf{V}|) \tag{38}$$

We thus see that there is no general-relativistic Magnus force for a VL motion with $|\mathbf{V}| = const$ around the core. This is possible only when the VL core accompanies the superfluid flow, i.e., when $\mathbf{V}$ is produced by the core itself.

Consequently, given Eq. (33), we see that there is no slip in general relativity either.

In conclusion, note that the absence of slip also follows from another important theorem of hydrodynamics, the theorem on the conservation of circulation (see, e.g., Landau and Lifshitz 1988a).

[9] The core radius is known to have sizes of the order of $1 \div 10$ interparticle separations.

5 Calculating the mean density and curvature of vortex lines in a pulsar with general-relativistic corrections

5.1 Passing to a Rotating Frame of Reference

As will be evident below, it is more convenient to perform an analysis in a comoving (with vortex cores), i.e., rotating frame of reference.

According to Landau and Lifshitz (1988b), the following formulas can be derived that relate the tensor components in various frames of reference:

$$\begin{cases} g'_{\rho\rho} = g_{\rho\rho}; \quad g'_{\varphi\varphi} = g_{\varphi\varphi}; \quad g'_{zz} = g_{zz} \\ g'_{0\varphi} = g_{0\varphi} + \frac{\Omega}{c} g_{\varphi\varphi}; \quad g'_{00} = g_{00} + (\frac{\Omega}{c})^2 g_{\varphi\varphi} + 2\frac{\Omega}{c} g_{0\varphi} \\ u'_\rho = u_\rho; \quad u'_\varphi = u_\varphi; \quad u'_z = u_z; \quad u'_0 = u_0 + \frac{\Omega}{c} u_\varphi. \end{cases} \tag{39}$$

Here, the components in a frame of references rotating with angular velocity Ω are marked by primes.

5.2 Calculating the Covariant Curl of Superfluid Velocity and the Vortex Density in the System

By definition, the mean density of vortices is the number of vortices crossing the orthogonal surface divided by its area. Therefore, using Eq. (22) to derive the vortex density $\sigma_{\beta\gamma}$, we obtain

$$\sigma_{\beta\gamma} = \frac{K}{dx^\beta \wedge dx^\gamma} = \frac{1}{2\pi}(\partial_\beta \phi_\gamma - \partial_\gamma \phi_\beta). \tag{40}$$

It is convenient to express the quantities ϕ_γ in terms of ϕ_0, the metric, and the shell angular velocity in the frame of reference under consideration:

$$\begin{cases} \phi_\alpha = \phi^0 g_{0\alpha} + \phi^\gamma g_{\alpha\gamma} = \phi^0 (g_{0\alpha} + g_{\alpha\gamma} \frac{\phi^\gamma}{\phi^0}), \\ \phi_0 = \phi^0 g_{00} + \phi^\gamma g_{0\gamma} = \phi^0 (g_{00} + g_{0\gamma} \frac{\phi^\gamma}{\phi^0}). \end{cases}$$

Hence, according to (33), we obtain for the average quantities

$$\hat{\phi}_\alpha = \hat{\phi}_0 \frac{g_{0\alpha} + \frac{\Omega}{c} g_{\varphi\alpha}}{g_{00} + \frac{\Omega}{c} g_{\varphi 0}}. \tag{41}$$

Since the average quantities do not depend on time and angle φ, we introduce the notation

$$X_\gamma = 2\pi \hat{\sigma}_{\varphi\gamma} = -\partial_\gamma \hat{\phi}_\varphi; \quad Y = \hat{\phi}_0; \quad Z = -\frac{g_{0\varphi} + \frac{\Omega}{c} g_{\varphi\varphi}}{g_{00} + \frac{\Omega}{c} g_{\varphi 0}}, \tag{42}$$

and derive from (41)

$$X_\gamma = \partial_\gamma (YZ). \tag{43}$$

On the other hand, we obtain from Eqs. (19) and (33)

$$X_\gamma = (c/\Omega)\partial_\gamma Y, \tag{44}$$

Solving the last two equations for X_γ and Y yields

$$X_\gamma = const\,\frac{\partial_\gamma Z}{(1-\frac{\Omega}{c}Z)^2}, \tag{45}$$

$$Y = \frac{const}{1-\frac{\Omega}{c}Z}. \tag{46}$$

Since $Y = \hat{\phi}_0 \to k \to m^* c/\hbar$ in the nonrelativistic limit, we see that $const = m^* c/\hbar$. Hence, we have

$$\hat{\sigma}_{\varphi\gamma} = \frac{m^* c}{2\pi\hbar}\frac{\partial_\gamma Z}{(1-\frac{\Omega}{c}Z)^2}. \tag{47}$$

As was shown above, $\Omega' = 0$ in the comoving (with cores) frame of reference; therefore, Eq. (47) in this frame of references is especially simple:

$$\boxed{\hat{\sigma^\alpha}' = \frac{m^* c}{2\pi\hbar\sqrt{\tilde{\gamma}'}} e^{\alpha\gamma\varphi}\partial_\gamma g'_\varphi, \qquad (48)}$$

where, according to Landau and Lifshitz (1988b), $g_\gamma = -g_{0\gamma}/g_{00}$, $\tilde{\gamma} = -g/g_{00}$ is the determinant of the spatial metric tensor, and the three-dimensional vector $\sigma^a lpha$ dual to the tensor $\sigma_{\gamma\beta}$ was defined as $\sigma^\alpha = (2\sqrt{\tilde{\gamma}})^{-1} \cdot e^{\alpha\gamma\beta}\sigma_{\gamma\beta}$, $e^{\alpha\gamma\beta}$ is a unit antisymmetric tensor. The vector $\hat{\sigma}^\alpha$ coincides in direction with the vortex direction in the system and is equal in magnitude to the mean vortex density at a given point.

For the invariant mean vortex density, we can write

$$\hat{\sigma} = \sqrt{\hat{\sigma}_{ij}\hat{\sigma}^{ij}} = \sqrt{2\hat{\sigma}_{0\alpha}\hat{\sigma}^{0\alpha} + \hat{\sigma}_{\alpha\beta}\hat{\sigma}^{\alpha\beta}} \tag{49}$$

As a result, we have for the invariant density in the first post-Newton approximation

$$\hat{\sigma} \approx |\hat{\sigma}_{\varphi\alpha}|\sqrt{g^{\alpha\alpha}g^{\varphi\varphi}}, \tag{50}$$

which matches the nonrelativistic limit for the VL density.

According to Eqs. (39), we have for g'_α

$$g'_\alpha = \delta^\varphi_\alpha \frac{g_\varphi - \frac{\Omega}{c}\frac{g_{\varphi\varphi}}{g_{00}}}{1 + (\frac{\Omega}{c})^2 \frac{g_{\varphi\varphi}}{g_{00}} - 2g_\varphi \frac{\Omega}{c}}. \qquad (51)$$

Given that $g' = g$, we obtain for $\tilde{\gamma}'$:

$$\tilde{\gamma}' = -\frac{g'}{g'_{00}} = \frac{\tilde{\gamma}}{1 + (\frac{\Omega}{c})^2 \frac{g_{\varphi\varphi}}{g_{00}} - 2\frac{\Omega}{c} g_{0\varphi}}.$$

In the nonrelativistic case, $\tilde{\gamma} \to \rho^2$, $g_{00} \to 1$, $g_{\varphi\varphi} \to -\rho^2$, $g_{zz} \to -1$, $g_{0\gamma} \to 0$, and $\phi_0 \to k \to m^* c/\hbar$; therefore, we derive the already known expression in this limit

$$\hat{\sigma}^\alpha \to \hat{\sigma}^\alpha_0 = \delta^\alpha_z \frac{\Omega m^*}{\pi\hbar} = const.$$

For a millisecond pulsar, $\hat{\sigma}^\alpha_0 \sim 10^{-6}\, cm^{-2}$, with a separation between vortices $d \sim 10^{-3}\, cm$ corresponding to this value. As for the radius of the core itself, its order-of-magnitude value is $10^{-11}\, cm$.

As we see from this section, the general-relativistic corrections that determine the curvature of VLs and the change in their mean density are small for real pulsars, and, therefore, the relative curvature does not exceed a few percent.

6 Calculating corrections for a homogeneous model.

Let us calculate the invariant density of vortex lines in a pulsar in the first post-Newton approximation. The model is based on the assumption that the pulsar interiors rotate at angular velocity Ω and, because the compressibility of a neutron condensate is low, its density is assumed to be $\tilde{\rho}$ in the entire volume [10].

In the first post-Newton approximation, the metric in a conformally Euclidean coordinate system [11] can be written as

$$ds^2 = (1 + 2\tilde{\varphi})dt^2 - (1 - 2\tilde{\varphi})[d\rho^2 + dz^2 + \rho^2 d\varphi^2] - 2g_\varphi dt\, d\varphi. \qquad (52)$$

According to Eq. (50), we write

$$\hat{\sigma} = \sqrt{g^{\gamma\gamma} g^{\varphi\varphi}} \frac{\hat{\sigma}_0}{2\Omega} \partial_\gamma [g_\varphi - \Omega g_{\varphi\varphi}/g_{00}], \qquad (\hat{\sigma}_0 = \frac{\Omega m^*}{\pi\hbar}). \qquad (53)$$

[10]For convenience of calculations, we take $G = 1$ and $c = 1$ in this section.)

[11]It is easy to see that the result does not depend on the choice of a coordinate system in this approximation.)

Since $g^{\alpha\alpha} = 1/g_{\alpha\alpha}$, the corrections to $\hat{\sigma}_0$ can be arbitrarily divided up into two groups:

(1) Gravimagnetic corrections:

$$\hat{\sigma}_1 = (\frac{m^*}{2\pi\hbar})\frac{(\partial_\rho + \partial_z)\, g_\varphi}{\rho}.$$

(2) Gravielectric corrections:

$$\hat{\sigma}_2 = \frac{\hat{\sigma}_0}{2}\frac{1+2\tilde{\varphi}}{\rho}(\partial_\rho + \partial_z)[\rho^2\frac{1-2\tilde{\varphi}}{1+2\tilde{\varphi}}] - \hat{\sigma}_0.$$

We thus see that the VL curvature and redistribution in a pulsar result from the gravimagnetic interaction of VLs with the Lense-Tirring field of the system and from the gravielectric deformation of the system's Euclidean geometry.

The Newton gravitational potential $\tilde{\varphi}$ of the model can be easily calculated:

$$\tilde{\varphi} = -2\pi\tilde{\rho}R^2(1 - x^2/3 - y^2/3) \qquad (x = \rho/R,\ y = z/R). \tag{54}$$

Hence, it is easy to derive an expression for the gravielectric corrections:

$$\hat{\sigma}_2 = 4\pi\tilde{\rho}R^2\hat{\sigma}_0(1 - x^2 - y^2/3 - 2xy/3).$$

Given that $\tilde{\varphi}_{_R} = -4\pi\tilde{\rho}R^2/3$ on the stellar surface, we obtain for the gravielectric corrections

$$\hat{\sigma}_2(x, y) = 3|\tilde{\varphi}_{_R}|\hat{\sigma}_0(1 - x^2 - y^2/3 - 2xy/3). \tag{55}$$

To determine the gravimagnetic corrections, we use Eq. (106.15) from Landau and Lifshitz (1988b) to derive the metric components g_φ. In Cartesian coordinates,

$$g_{0\alpha}(\mathbf{r}) = \frac{1}{2}\int_V \tilde{\rho}\, dr'^3\{\frac{7[\mathbf{\Omega}\times\mathbf{r}']_\alpha + ([\mathbf{\Omega}\times\mathbf{r}']_\beta\,, n_\beta)\, n_\alpha}{|\mathbf{r}-\mathbf{r}'|}\}, \tag{56}$$

where $n_\alpha = (r_\alpha - r'_\alpha)/|\mathbf{r}-\mathbf{r}'|$.

Hence, it is easy to calculate the angular component of the gravimagnetic field in cylindrical coordinates:

$$g_\varphi(\mathbf{r}) = -\frac{\rho}{2}\int_V \tilde{\rho}\, d\varphi|^{+\pi}_{-\pi}dz'\rho' d\rho'\{\frac{7\Omega\rho'\cos\varphi}{|\mathbf{r}-\mathbf{r}'|} + \frac{\Omega\rho' p^2\cos\alpha\cos\gamma}{|\mathbf{r}-\mathbf{r}'|^3}\}, \tag{57}$$

Here, $(\mathbf{r}-\mathbf{r}')^2 = (z-z')^2 + p^2, \quad p^2 = \rho^2 + {\rho'}^2 - 2\rho\rho'\cos\varphi,$
$\cos\alpha = \cos\varphi\sin\theta + \sin\varphi\cos\theta, \quad \cos\gamma = -\sin\theta, \quad \sin\theta = \frac{\rho'}{p}\sin\varphi,$ and

$\cos\theta = \frac{p^2+\rho^2-\rho'^2}{2p\rho}$.
As a result, expression (57) reduces to

$$g_\varphi(\mathbf{r}) = -\frac{\Omega\rho}{2}\int\limits_V \tilde{\rho}\, d\varphi|_{-\pi}^{+\pi} dz'\rho' d\rho' \{\frac{7\rho'\cos\varphi}{|\mathbf{r}-\mathbf{r}'|} + \frac{\rho'\sin^2\varphi(\rho'^2+p\rho-p\rho'\cos\varphi)}{|\mathbf{r}-\mathbf{r}'|^3}\}. \tag{58}$$

Since the integrand is even in variable φ, by changing variables: $x = \rho/R$, $x' = \rho'/R$, $y = z/R$, $y'(y) = (z'-z)/R$, and $p'(x,\varphi) = p/R$, we can write the expression for the gravimagnetic corrections as

$$\hat{\sigma}_1(x,y) = \hat{\sigma}_0|\tilde{\varphi}_{R}| \int\limits_0^\pi d\varphi \int\limits_0^1 dx' f(x,x',y,\varphi) \tag{59}$$

where

$$f(x,x',y,\varphi) = -\tfrac{3x'^2}{8\pi x}(\partial_x+\partial_y)\int\limits_{y_1'}^{y_2'} dy' \left\{\tfrac{7x\cos\varphi}{[y'^2+p'^2]^{1/2}} + \tfrac{xx'\sin^2\varphi(x'^2+p'x-p'x'\cos\varphi)}{[y'^2+p'^2]^{3/2}}\right\},$$

$$y_1' = -\sqrt{1-x'^2} - y, \quad y_2' = +\sqrt{1-x'^2} - y. \tag{60}$$

Hence, integrating and then differentiating yields

$$\begin{aligned}
&A = 7x\cos\varphi,\\
&B = xx'\sin^2\varphi(x'^2+p'x-p'x'\cos\varphi),\\
&C = \sqrt{y'^2+p'^2}\\
&A_x = 7\cos\varphi,\\
&B_x = x'\sin^2\varphi(x'^2+p'x-p'x'\cos\varphi) + xx'\sin^2\varphi(p'+(x-x'\cos\varphi)^2/p'),\\
&I_x = A_x\ln(y'+C) + A\tfrac{x-x'\cos\varphi}{C(y'+C)} + \tfrac{B_x y'}{Cp'^2} - By'\tfrac{x-x'\cos\varphi}{p'^2C^3} - 2By'\tfrac{x-x'\cos\varphi}{p'^4C},\\
&I_y = -A/C - B/C^3,\\
&\Longrightarrow f(x,x',y,\varphi) = -\tfrac{3x'^2}{8\pi x}[I_x+I_y]_{y_1'}^{y_2'}.
\end{aligned}$$

When deriving the last expression, we took into account the fact that the derivative with respect to y could be taken inside the integral and that $\partial_y = -\partial_{y'}$.

Integral (59) can be calculated numerically. The integration results with gravielectric corrections (55) are shown in the figure (the maximum amplitude of relative corrections is ≈ 1).

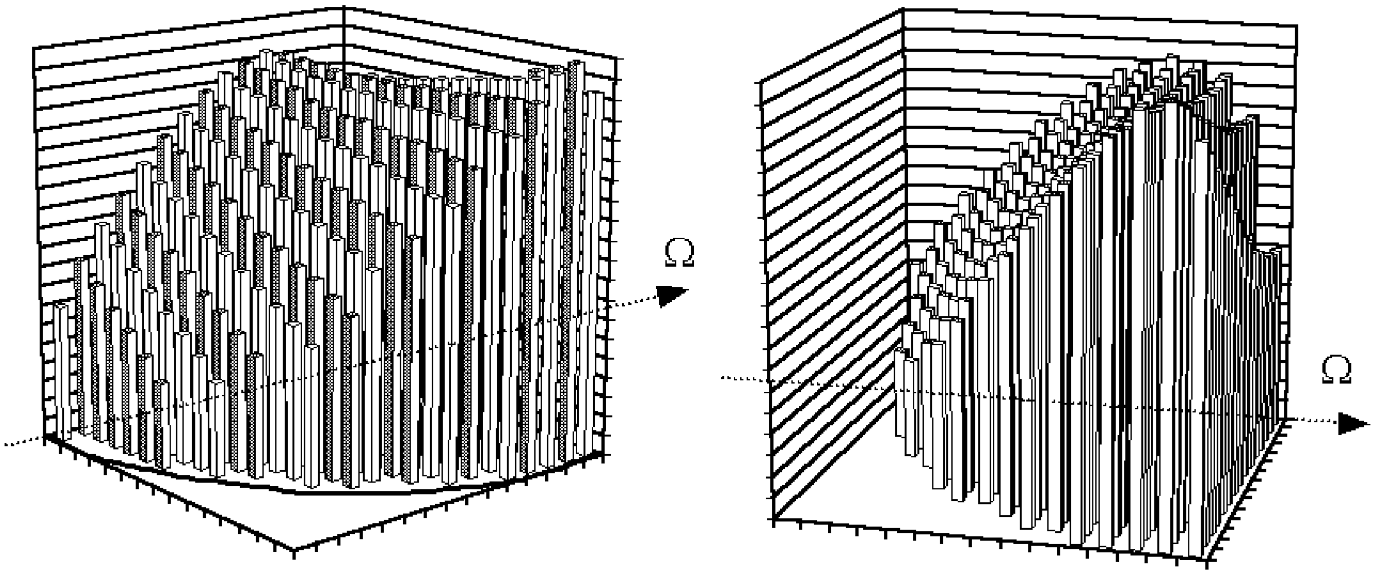

Fig. 1: Dependence of corrections $(\hat{\sigma}_1+\hat{\sigma}_2)/(\hat{\sigma}_0|\tilde{\varphi}_R|)$ along the vertical axis on coordinates: the rotation axis Ω and the perpendicular axis in the equatorial plane (the lower right quadrant of the pulsar meridional section is shown).

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